Lecture 21. Methods for solving the radiation transfer equation.
Part 4: Principles of invariance. Adding method.

Objectives:

1. Principles of invariance.
2. Adding method.

Required reading:

L02: 6.3.1- 6.3.4, 6.4

1. Principles of invariance

Recall the definitions of reflection and transmission of a layer introduced in Lec.18-19.

If solar flux is incident on a layer of optical depth **τ***:

$$R(\mu, \varphi, \mu_0, \varphi_0) = \pi I^{\uparrow}(0, \mu, \varphi) / \mu_0 F_0$$

$$T(\mu, \varphi, \mu_0, \varphi_0) = \pi I^{\downarrow}(\tau^*, -\mu, \varphi) / \mu_0 F_0$$

General case:

$$I_r^{\uparrow}(0, \mu, \varphi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 R(\mu, \varphi, \mu', \varphi') I_{inc}(-\mu', \varphi')\mu' d\mu' d\varphi'$$

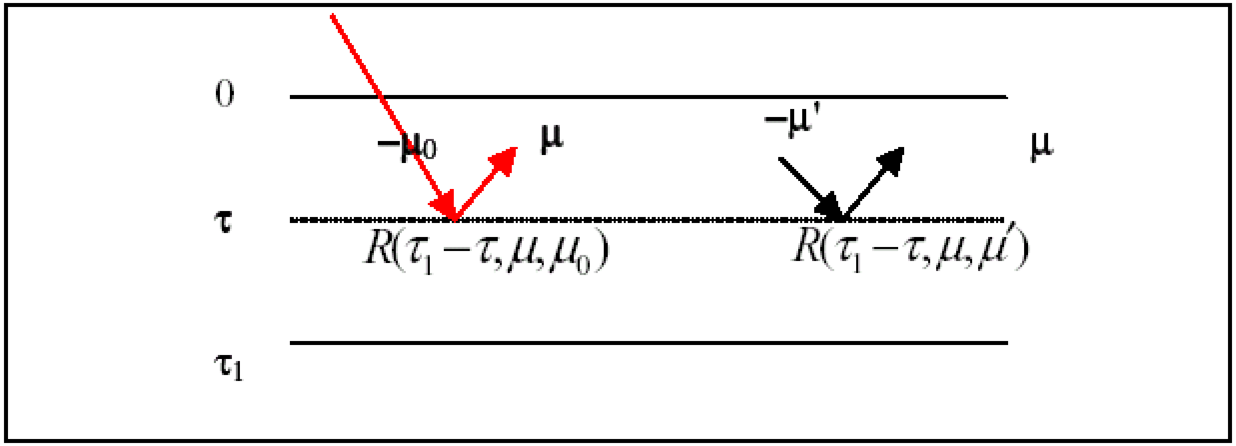
$$I_t^{\downarrow}(\tau^*, -\mu, \varphi) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 T(\mu, \varphi, \mu', \varphi') I_{inc}(-\mu', \varphi')\mu' d\mu' d\varphi'$$

- **The principle of invariance for the semi-infinite atmosphere** (Ambartzumian, 1940): the diffuse reflected intensity cannot be changed if a layer of finite optical depth, having the same optical properties as those of the original layer, is added (see L02: 6.3.2).

➢ **The principles of invariance for the <u>finite</u> atmosphere** (Chandrasekhar, 1950):

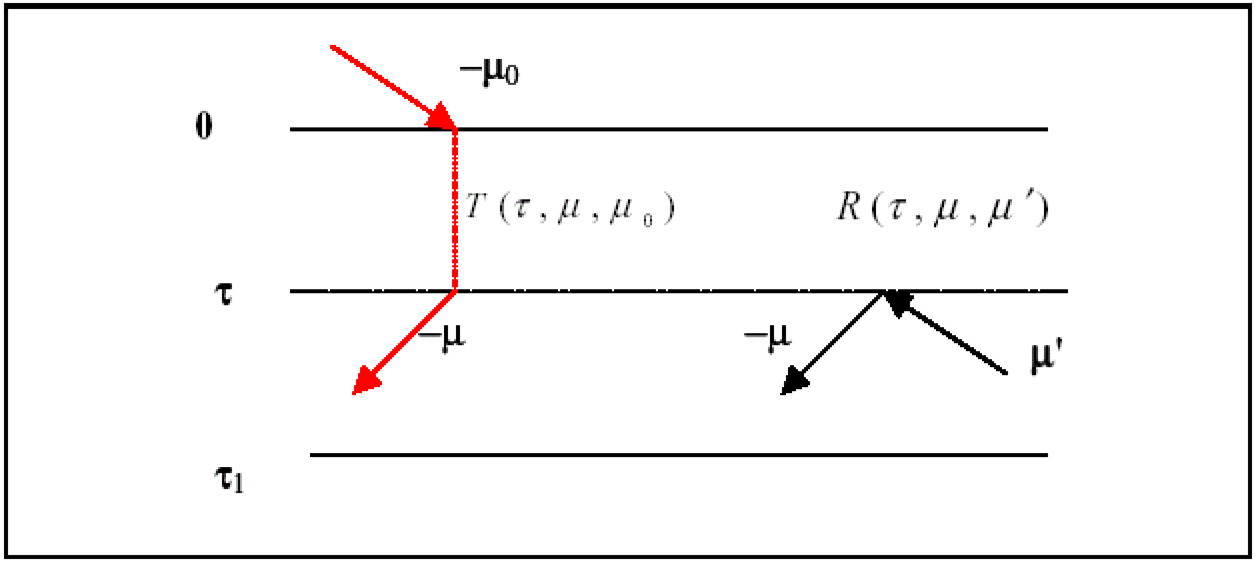
(1) The reflected (upward) intensity at any given optical depth **τ** results from the reflection of (a) the attenuated solar flux = $\mu_0 F_0 \exp(-\tau/\mu_0)$ and (b) the downward diffuse intensity at the level **τ:**

$$I^{\uparrow}(\tau,\mu) = \frac{\mu_0 F_0}{\pi} \exp(-\tau/\mu_0) R(\tau_1-\tau,\mu,\mu_0) + 2\int_0^1 R(\tau_1-\tau,\mu,\mu') I^{\downarrow}(\tau,-\mu')\mu' d\mu' \quad [21.1]$$



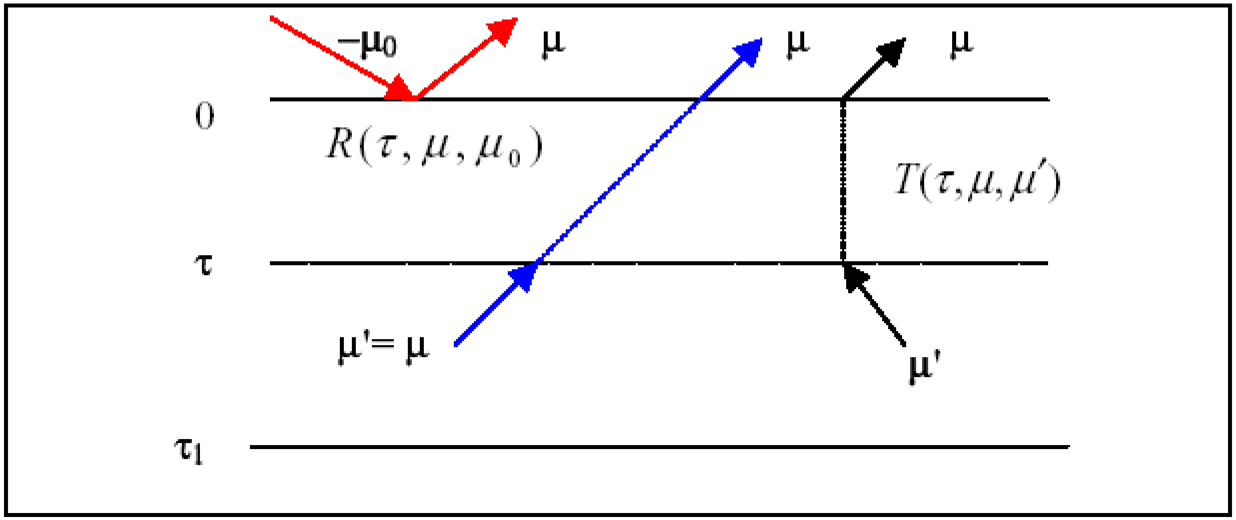
(2) The diffusely transmitted (downward) intensity at the level $\boldsymbol{\tau}$ results from (a) the transmission of incident solar flux and (b) the reflection of the upward diffuse intensity above the level $\boldsymbol{\tau}$:

$$I^{\downarrow}(\tau,-\mu)=\frac{\mu_0 F_0}{\pi}T(\tau,\mu,\mu_0)+2\int_0^1 R(\tau,\mu,\mu')I^{\uparrow}(\tau,\mu')\mu' d\mu' \qquad [21.2]$$



(3) The reflected (upward) intensity at the top of the finite atmosphere ($\boldsymbol{\tau} = 0$) is equivalent to (a) the reflection of solar flux plus (b) the direct and diffuse transmission of the upward diffuse intensity above the level $\boldsymbol{\tau}$:

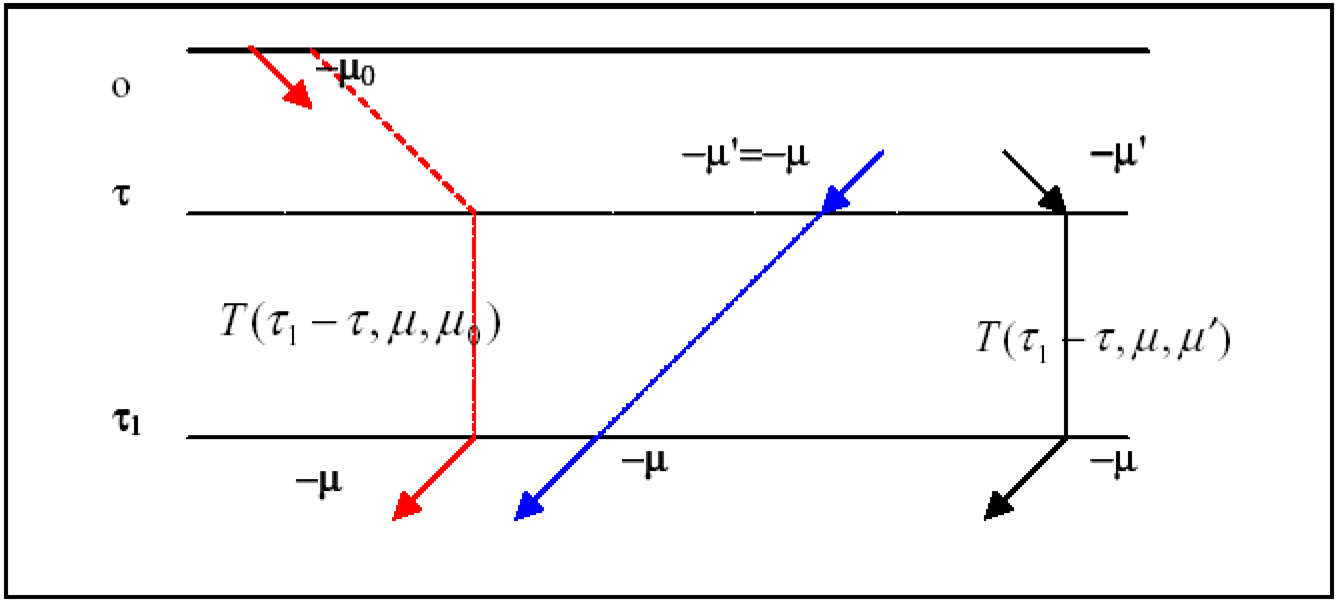
$$I^{\uparrow}(0,\mu) = \frac{\mu_0 F_0}{\pi} R(\tau,\mu,\mu_0) + 2\int_0^1 T(\tau,\mu,\mu') I^{\uparrow}(\tau,\mu')\mu' d\mu' + I^{\uparrow}(\tau,\mu)\exp(-\tau/\mu) \qquad [21.3]$$



(4) The diffusely transmitted (downward) intensity at the bottom of the finite atmosphere ($\tau=\tau_1$) is equivalent to (a) the transmission of the attenuated solar flux at the level τ plus (b) the direct and diffuse transmission of the downward diffuse intensity at the level τ from above:

$$I^{\downarrow}(\tau_1,-\mu) = \frac{\mu_0 F_0}{\pi}\exp(-\tau/\mu_0)T(\tau_1-\tau,\mu,\mu_0) + 2\int_0^1 T(\tau_1-\tau,\mu,\mu')I^{\downarrow}(\tau,-\mu')\mu' d\mu' + \qquad [21.4]$$

$$+ I^{\downarrow}(\tau,-\mu)\exp(-(\tau_1-\tau)/\mu)$$



2. Adding method

Adding method is an “exact” technique for solving the radiative transfer equation with multiple scattering. It uses geometrical ray-tracing approach and the reflection and transmission of each individual atmospheric layer.

Strategy: knowing the reflection and transmission of two individual layers, the reflection and transmission of the combined layer may be obtained by calculating the successive reflections and transmissions between these two layers.

NOTE: If optical depths of these two layers are equaled, this method is referred to as the doubling-adding method.

Multi-Stream Radiative Transfer

The two major numerical plane-parallel radiative transfer methods both use discrete ordinates and give the same numerical results.

1. Start with the RTE Fourier transformed in azimuth.
2. Replace the scattering integral by a quadrature sum.
3. The RTE becomes a ordinary differential *matrix* equation.

Radiative transfer equation for each Fourier azimuthal mode m:

$$\begin{aligned}
\mu \frac{dI_m(\tau,\mu)}{d\tau} &= I_m(\tau,\mu) - \frac{\omega}{2}\sum_{l=m}^{N} a_{lm}\omega_l \mathcal{P}_l^m(\mu)\int_{-1}^{+1}\mathcal{P}_l^m(\mu')I_m(\tau,\mu')d\mu' \\
&\quad -\frac{\omega}{4\pi}\sum_{l=m}^{N} a_{lm}\omega_l \mathcal{P}_l^m(\mu)\mathcal{P}_l^m(-\mu_0)S_0 e^{-\tau/\mu_0}
\end{aligned}$$

Discrete Ordinates and Quadrature Sums

An integral may be approximated by a quadrature sum

$$\int_{-1}^{1} f(\mu) d\mu \approx \sum_{j=1}^{N} w_j f(\mu_j)$$

where μ_j are the *discrete ordinates* and w_j are quadrature weights.

In Gaussian quadrature, μ_j are roots of Legendre polynomial $\mathcal{P}_N(\mu)$. Weights are $w_j = 2/\{(1-\mu_j^2)[\mathcal{P}_N'(\mu_j)]^2\}$. Weights sum to unity.

Gaussian quadrature is exact for polynomials up to degree $2N-1$.

For plane-parallel RT, double-Gauss quadrature is more accurate. Double-Gauss: separate quadrature sum for each hemisphere, e.g.

$$\int_{0}^{1} I^{\uparrow}(\mu) d\mu + \int_{-1}^{0} I^{\downarrow}(\mu) d\mu \approx \sum_{j=1}^{N} w_j I^{\uparrow}(\mu_j) + \sum_{j=1}^{N} w_j I^{\downarrow}(-\mu_j)$$

where μ_j are Gaussian angles scaled from (-1,+1) to (0,+1).

Discrete Ordinates Radiative Transfer Equation

Discrete ordinates: $2N$ streams, N upward and N downward.

Discrete ordinate RTE

$$\begin{aligned}\pm\mu_j \frac{dI_m(\tau,\pm\mu_j)}{d\tau} &= I_m(\tau,\pm\mu_j) - \frac{\omega}{2}\left[\sum_{j'=1}^{N} w_{j'} P_m^{\pm +}(\pm\mu_j,\mu_{j'}) I_m(\mu_{j'})\right. \\ &\left. + \sum_{j'=1}^{N} w_{j'} P_m^{\pm -}(\pm\mu_j,-\mu_{j'}) I_m(-\mu_{j'})\right] + \mathcal{S}(\pm\mu_j)\end{aligned}$$

The m'th Fourier mode of the phase function is

$$P_m^{\pm +}(\pm\mu_j,\mu_{j'}) = \sum_{l=m}^{N} (2-\delta_{0,m}) \frac{(l-m)!}{(l+m)!} \omega_l \mathcal{P}_l^m(\pm\mu_j) \mathcal{P}_l^m(\mu_{j'})$$

The pluses refer to upwelling, and the minuses to downwelling.

Matrix Form of Radiative Transfer Equation

Radiances at discrete angles are up and down vectors $(\mathbf{I}^+, \mathbf{I}^-)$.

The matrix RTE is

$$M\frac{d}{d\tau}\begin{pmatrix}\mathbf{I}^+\\ \mathbf{I}^-\end{pmatrix} = \begin{pmatrix}\mathbf{I}^+\\ \mathbf{I}^-\end{pmatrix} - \begin{pmatrix}P^{++} & P^{+-}\\ P^{-+} & P^{--}\end{pmatrix}\begin{pmatrix}\mathbf{I}^+\\ \mathbf{I}^-\end{pmatrix} - \begin{pmatrix}\mathcal{S}^+\\ \mathcal{S}^-\end{pmatrix}$$

I is radiance vector (one hemisphere of μ_j, one Fourier mode m)
$\mathcal{S}$ is source vector (e.g. diffuse "pseudo-source")
M is diagonal matrix with $\pm\mu_j$
P is discrete ordinate phase function matrix (with $\omega/2$ and weights w_j).

Reciprocity principle: $P^{++} = P^{--} \quad P^{+-} = P^{-+}$

P^{++} is phase function for upwelling incident and upwelling scattered directions.

P^{+-} is phase function for downwelling incident and upwelling scattered.

Interaction Principle

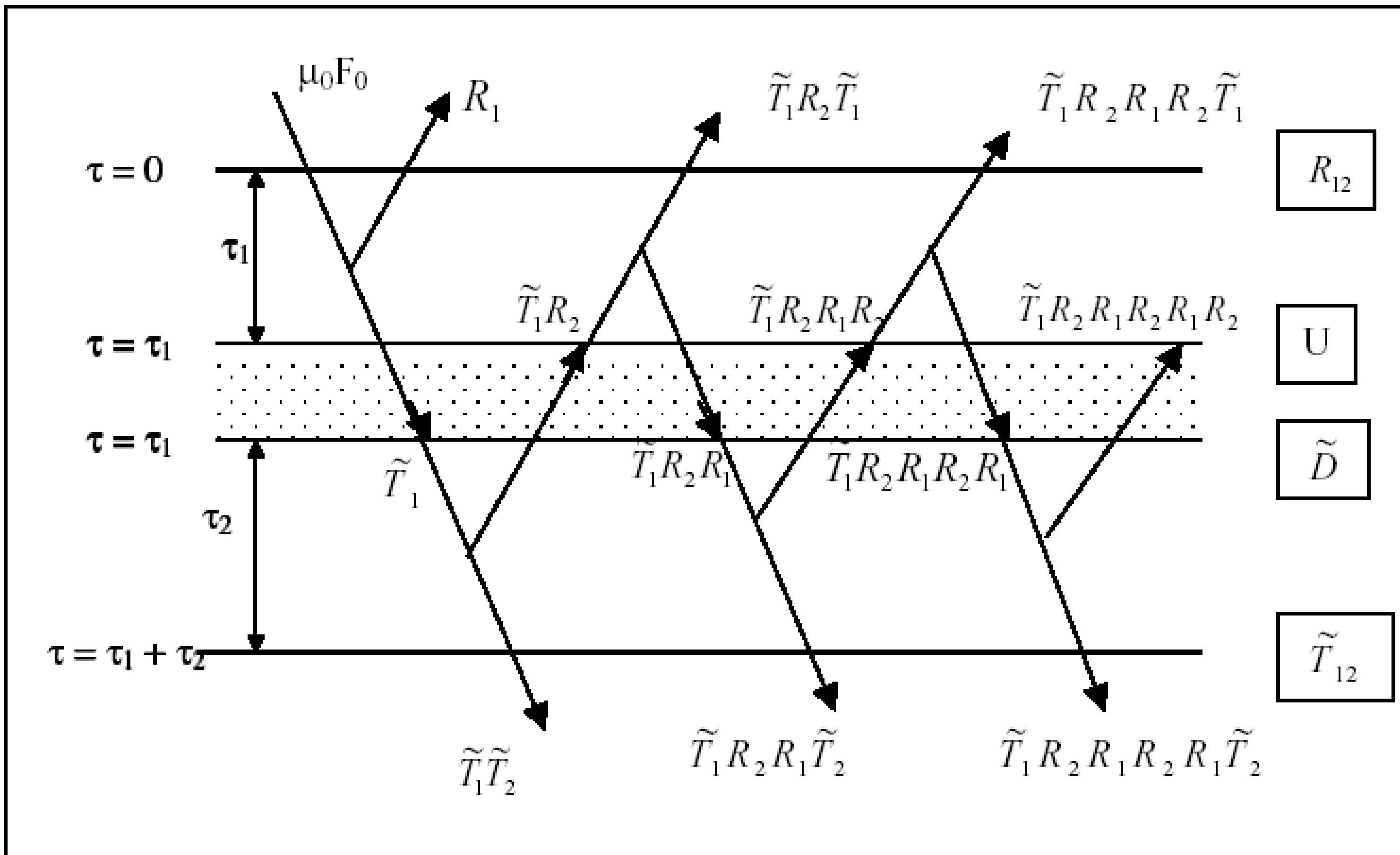
The RTE is a linear equation in radiance: radiance exitting a layer is linear in radiance incident upon the layer. Represent radiative transfer in a layer with matrix equations.

Interaction principle:

$$I_0^+ = T^+ I_1^+ + R^+ I_0^- + S^+ \qquad I_1^- = T^- I_0^- + R^- I_1^+ + S^-$$

I_0^- and I_1^+ are incident radiance vectors,
T is the transmission matrix,
R is the reflection matrix,
S is the source vector (solar pseudo-source or thermal emission)

Consider two layers with reflection R_1 and R_2 and total (direct plus diffuse) transmission $\tilde{T}_1$ and $\tilde{T}_2$ functions, respectively. Let's denote the combined reflection and total transmission functions by R_{12} and $\tilde{T}_{12}$, and combined reflection and total transmission functions between layers 1 and 2 by U and $\tilde{D}$, respectively.



The combined reflection function R_{12} is

$$
\begin{aligned}
R_{12} &= R_1 + \widetilde{T}_1 R_2 \widetilde{T}_1 + \widetilde{T}_1 R_2 R_1 R_2 \widetilde{T}_1 + \widetilde{T}_1 R_2 R_1 R_2 R_1 R_2 \widetilde{T}_1 + \ldots = \\
&= R_1 + \widetilde{T}_1 R_2 \widetilde{T}_1 [1 + R_1 R_2 + (R_1 R_2)^2 + \ldots] = \\
&= R_1 + R_2 \widetilde{T}_1^{\,2} (1 - R_1 R_2)^{-1}
\end{aligned}
\qquad [21.5]
$$

NOTE: In Eq.[21.5] we use that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

The combined total transmission function $\widetilde{T}_{12}$ is

$$
\begin{aligned}
\widetilde{T}_{12} &= \widetilde{T}_1 + \widetilde{T}_1 R_2 R_1 \widetilde{T}_2 + \widetilde{T}_1 R_2 R_1 R_2 R_1 \widetilde{T}_2 + \ldots = \\
&= \widetilde{T}_1 \widetilde{T}_2 [1 + R_1 R_2 + (R_1 R_2)^2 + \ldots] = \\
&= \widetilde{T}_1 \widetilde{T}_2 (1 - R_1 R_2)^{-1}
\end{aligned}
\qquad [21.6]
$$

The combined reflection function U between layers 1 and 2:

$$U = \widetilde{T}_1 R_2 + \widetilde{T}_1 R_2 R_1 R_2 + \widetilde{T}_1 R_2 R_1 R_2 R_1 R_2 + \ldots =$$
$$= \widetilde{T}_1 R_2 [1 + R_1 R_2 + (R_1 R_2)^2 + \ldots] = \qquad [21.7]$$
$$= \widetilde{T}_1 R_2 (1 - R_1 R_2)^{-1}$$

The combined total transmission function $\widetilde{D}$ between layers 1 and 2:

$$\widetilde{D} = \widetilde{T}_1 + \widetilde{T}_1 R_2 R_1 + \widetilde{T}_1 R_2 R_1 R_2 R_1 + \ldots =$$
$$= \widetilde{T}_1 [1 + R_1 R_2 + (R_1 R_2)^2 + \ldots] = \qquad [21.8]$$
$$= \widetilde{T}_1 (1 - R_1 R_2)^{-1}$$

From Eqs.[21.5]-[21.8], we find that

$$R_{12} = R_1 + \widetilde{T}_1 U \;; \;\; \widetilde{T}_{12} = \widetilde{T}_2 \widetilde{D} \;; \;\; U = R_2 \widetilde{D} \qquad [21.9]$$

Let's introduce $S = R_1 R_2 (1 - R_1 R_2)^{-1}$

Using that $\tilde{T} = T + \exp(-\tau / \mu')$, from Eqs.[21.8]-[21.9] we find

$$\begin{aligned} \tilde{D} &= D + \exp(-\tau_1 / \mu_0) = \\ &= (1+S)(T_1 + \exp(-\tau_1 / \mu_0)) = (1+S)T_1 + S\exp(-\tau_1 / \mu_0) + \exp(-\tau_1 / \mu_0) \end{aligned} \qquad [21.10]$$

$$\begin{aligned} \tilde{T}_{12} &= (T_2 + \exp(-\tau_2 / \mu_0))(D + \exp(-\tau_1 / \mu_0)) \\ &= D\exp(-\tau_2 / \mu_0) + T_2 \exp(-\tau_1 / \mu_0) + T_2 D + \exp\left(-\left[\frac{\tau_1}{\mu_0} + \frac{\tau_2}{\mu}\right]\right)\delta(\mu - \mu_0) \end{aligned} \qquad [21.11]$$

Thus, we may write a system of iterative equations for the computation of diffuse transmission and reflection for the two layers in the form:

$$\boxed{\begin{aligned} &Q = R_1 R_2 \\ &S = Q(1-Q)^{-1} \\ &D = T_1 + ST_1 + S\exp(-\tau_1/\mu_0) \\ &U = R_2 D + R_2 \exp(-\tau_1/\mu_0) \\ &R_{12} = R_1 + \exp(-\tau_1/\mu)U + T_1 U \\ &T_{12} = \exp(-\tau_2/\mu)D + T_2\exp(-\tau_1/\mu_0) + T_2 D \end{aligned}} \qquad [21.12]$$

NOTE: in Eq.[21.12], the product of two functions implies n integration over the appropriate angle so that all multiple-scattering contributions are included. For instance

$$R_1 R_2 = 2\int_0^1 R_1(\mu,\mu')R_2(\mu',\mu_0)\mu' d\mu'$$

Numerical procedure of the adding method:

1) As the starting point, one may calculate the reflection and transmission functions of an initial layer of very small optical depth (e.g., $\boldsymbol{\Delta\tau} = 10^{-8}$) that the single scattering approximation is applicable.

2) Then, using Eq.[21.12], one computes the reflection and transmission functions of the layer of 2 $\boldsymbol{\Delta\tau}$.

3) Using Eq.[21.12], one repeats the calculations adding the layers until a desirable optical depth is achieved.

Adding-Doubling method.

Initialization

Initialization: get R,T,S properties of infinitesimal layer from matrix RTE.

Use finite difference of RTE (optically thin solution) to get properties for layer optical depth $\delta\tau$ (e.g. $\delta\tau = 10^{-5}$).

$$R^{+} = \delta\tau M^{-1} P^{+-} \quad T^{+} = 1 - \delta\tau M^{-1}(1 - P^{++}) \quad S^{+} = \delta\tau M^{-1} \mathcal{S}^{+}$$

Adding Formula

The properties (R and T matrices, S vector) for a combination of two layers can be found from the interaction principle.

The adding formula can be derived from multiple reflections:

$$\begin{aligned}
R_T &= R_1 + T_1 R_2 T_1 + T_1 R_2 R_1 R_2 T_1 + T_1 R_2 R_1 R_2 R_1 R_2 T_1 + \ldots \\
R_T &= R_1 + T_1 R_2 \left[1 + R_1 R_2 + R_1 R_2 R_1 R_2 + \ldots\right] T_1 \\
R_T &= R_1 + T_1 R_2 \left[1 - R_1 R_2\right]^{-1} T_1
\end{aligned}$$

Adding formulas for upwelling radiance (similar for downwelling):

$$R_T^{+} = R_1^{+} + T_1^{+} \Gamma^{+} R_2^{+} T_1^{-} \qquad T_T^{+} = T_1^{+} \Gamma^{+} T_2^{+}$$

$$S_T^{+} = S_1^{+} + T_1^{+} \Gamma^{+} (S_2^{+} + R_2^{+} S_1^{-})$$

$$\Gamma^{+} = [1 - R_2^{+} R_1^{-}]^{-1}$$

Γ is a multiple reflection factor.

Doubling

Doubling: using adding formulas on identical layers.

Doubling formula

$$R^+_{2n\delta\tau} = R^+_{n\delta\tau} + T^+_{n\delta\tau}\Gamma^+ R^+_{n\delta\tau} T^-_{n\delta\tau} \qquad T^+_{2n\delta\tau} = T^+_{n\delta\tau}\Gamma^+ T^+_{n\delta\tau}$$

$$\Gamma^+ = [1 - R^+_{n\delta\tau} R^-_{n\delta\tau}]^{-1}$$

For exponential in optical depth source function:

$$S^+_{2n\delta\tau} = S^+_{n\delta\tau} + T^+_{n\delta\tau}\Gamma^+(\gamma^n S^+_{n\delta\tau} + R^+_{n\delta\tau} S^-_{n\delta\tau})$$

$$S^-_{2n\delta\tau} = \gamma^n S^-_{n\delta\tau} + T^-_{n\delta\tau}\Gamma^-(S^-_{n\delta\tau} + R^-_{n\delta\tau} S^+_{n\delta\tau}\gamma^n)$$

where $\gamma = \exp(-\delta\tau/\mu_0)$.

After N doubling steps optical thickness is $2^N\delta\tau$.
What is optical depth after $N = 20$ doubling steps?

Doubling-Adding Method

1. Addition theorem used to calculate Fourier transformed phase function $P_m^{\pm\pm}(\pm\mu_j, \pm\mu_{j'})$ at quadrature angles.

2. Initialization: local reflection R and transmission T matrices for initial layer $\delta\tau$ made from phase function, etc.

3. Doubling: use doubling formula n times to get R, T, S for homogeneous layer with $\Delta\tau = 2^n\delta\tau$.

4. Adding: use adding formula to combine distinct homogeneous layers and surface together (surface has $T = 1, R = R_s, S = 0$).

5. Use interaction principle to apply boundary conditions and obtain outgoing discrete ordinate radiances (or internal radiances).